Air and Space this Week

Item of the Week

The Mathematics of a Geostationary Orbit

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October 1 is the 75th anniversary of the publication of a paper by science (fiction) writer Arthur C. Clarke about placing communications relays in geostationary orbit in order to facilitate global "over the horizon" radio communications. The mathematical description of a geostationary orbit, the theme of this Item, was derived from Newton's and Kepler's Laws literally centuries ago, but it was Clarke who first advocated for the exploitation of this valuable location in Space. [Clarke's visionary foresight, however, did not extend to possible advances in electronics. His relay satellites would be Space Stations, because they needed to house the dozens of technicians and their logistical support needed to maintain the thousands of vacuum tubes used in the relays!] Today, much of our routine TV, radio, financial transactions, and more, depends on using relay satellites in geostationary orbits.

This particular anniversary is a good chance to share with others the tremendous economic and quality-of-life advances, such as the exploitation of geosynchronous orbit enabled by NASA-developed technology, have had for all of us. I wanted to give you a review of the math behind geosynchronicity to help you with that conversation.

Definitions:

An object is in *geosynchronous orbit* around the Earth if its orbital period is equal to one day. From the ground, the satellite wouldn't rise or set, but, depending on its orbit inclination and eccentricity, it would move around somewhat. Accommodating such movement would make the receiver tracking system more complicated than if the satellite were In…

A *geostationary orbit* around the Earth is both circular and in the plane of Earth's Equator (called the Ecliptic Plane). In such a case, the satellite appear from the ground to remain in the same spot in the sky. A receiving antenna on the ground would not have to move in order to send/receive signals from a geostationary satellite, greatly simplifying the antenna mounting and motion requirements.

Approach: We will use Newton's formulation for the (centripetal) force required to cause a satellite to have a circular orbit, and then find the orbit size that would allow for the satellite to be geostationary. Along the way, we'll derive Kepler's Third Law, and see how to use it to determine masses of objects far from Earth.

Start: The generalized equation for centripetal force associated with uniform circular motion is:

$$F_{centripital} = m \times a_{centripetal} = (m \times v^2)/r$$

where: m is the mass of the satellite
v is the speed of the satellite and
r is the radius of the circular orbit
"x" will represent multiplication in this notation

Since the centripetal force is due to Earth's gravity, *a la* Newton:

$$F_{centripetal} = (G \times M \times m) / r^2$$

where: G is the Universal Gravitational Constant
M is the mass of the Earth

Equating the two:

$$(m \times v^2) / r = (G \times M \times m) / r^2$$

Multiplying both sides by r and dividing both sides by m gives:

$$v^2 = (G \times M) / r$$

Now, let's consider what v represents in a circular orbit. The circumference of a circular orbit is the distance an orbiting satellite has to traverse to make one orbit:

$$\text{Circumference} = 2 \times \pi \times r$$

And the time it takes for that satellite to complete one orbit, the *orbit period*, P, a simple distance traveled versus speed situation, is:

$$v = (2 \times \pi \times r) / P$$

Square both sides:

$$v^2 = (4 \times \pi^2 \times r^2) / P^2$$

Equate the two expressions for v^2 above:

$$(G \times M) / r = (4 \times \pi^2 \times r^2) / P^2$$

Multiply both sides by r:

$$(G \times M) = (4 \times \pi^2 \times r^3) / P^2$$

Multiply both sides by P^2 and divide both sides by (G x M):

$$P^2 = ((4 \times \pi^2) / (G \times M)) \times r^3$$

This is Kepler's Third Law, usually stated as "the square of the period of a satellite is proportional to the cube of the radius of its orbital radius." Recall that this is for a satellite in circular orbit. The equation is valid for elliptical orbits, too, but instead of using r, the radius of a circular orbit, you'd use the ellipse's semi-major axis. The constant of proportionality is independent of all variables, except the mass of the Earth (or any body whose gravity is enabling the orbit), and is equal to: $((4 \times \pi^2) / (G \times M))$.

This is an extraordinarily-powerful equation! It allows us to not only calculate the height needed for a geostationary orbit, it also allows us to determine the mass of any object being orbited, be it exoplanet and star, planet and moon, etc.

So, how high does an orbit have to be for the satellite to be geostationary?

Let's re-arrange Kepler's Third Law and solve for r:

Start with: $P^2 = ((4 \times \pi^2) / (G \times M)) \times r^3$ from above

Divide both sides by $(4 \times \pi^2)$ and multiply both sides by (G x M):

$$((G \times M) / (4 \times \pi^2)) \times P^2 = r^3$$

Now take the cube root of both sides:

$$r = \text{cube root } (((G \times M) / (4 \times \pi^2)) \times P^2)$$

G and M are known quantities, as is π. All we need to do is to plug in a value for P and we can calculate the necessary orbit height. But first, we have to realize two important things.

1. The value of r is the height of the satellite above the center of the Earth, not the height of the satellite above the surface of the Earth. To get that, the radius of the Earth below the satellite has to be subtracted from the value for r.
2. There are two kinds of days: solar and sidereal. The *sidereal day* is the time it takes for the Earth to turn once on its axis relative to the stars. A *solar day* is the time it takes for the Earth to turn once on its axis PLUS a little bit more to accommodate the Earth's movement around the Sun during that particular day. The solar day is about four minutes longer than the sidereal day, but it's the sidereal day's length we have to use here.

When all the values are plugged in, the height of a geosynchronous orbit is found to be ~35,786 km (22,236 miles). There is a slight variation due to the Earth not being a perfect sphere.

The orbit mechanics described above has been known literally for centuries. But it wasn't until line-of-sight radio communications and rocketry developed sufficiently before we could contemplate launching to exploit this very special orbit, and it was Arthur Clarke leading the way. His first papers on the topic date from 1945; the first geostationary orbit was achieved by *Syncom 3* in 1964.

This Wikipedia article has a good summary of geostationary orbit utilization. And just to show you I leave no stone unturned in the search for good explanations, you can find a good synopsis of the calculations above here.

But WAIT, there's more!

There is another way to re-arrange the equation from above:

$$(m \times v^2) / r = (G \times M \times m) / r^2$$

Let's solve for M, instead for r. Divide both sides by m and multiply both sides by r:

$$v^2 = (G \times M) / r$$

Recall from above that $v = (2 \times \pi \times r) / P$

Square both sides:

$$v^2 = (4 \times \pi^2 \times r^2) / P^2$$

Equate both expressions for v^2 and solve for M:

$$((G \times M) / r) = ((4 \times \pi^2 \times r^2) / P^2)$$

Multiply both sides by r:

$$(G \times M) = ((4 \times \pi^2 \times r^3) / P^2)$$

Divide both sides by G:

$$M = (4 \times \pi^2 \times r^3) / (G \times P^2)$$

This equation, too, is a powerful tool. G and π are fundamental numbers, so if we can measure r and P for an orbiting system, then the old plug-and-chug yields the value for M. Think about that for a moment.

If we look at a moon orbiting a planet, the ***only things*** we need to know to determine the mass of that planet is the period of the moon's orbit and the size of the orbit's semi-major axis. That's a straightforward measurement to make for objects within our Solar System, but it works for more distant objects, too (*e.g.* planets around other stars, and even a satellite galaxy around the galaxy it orbits).

For example, if we can make detailed observations of a binary star, especially those where one star is much less-massive than the other, then the mass of the larger star can be calculated from movements of the smaller (the size and period of its orbit).

And Yet More!

Recall from above:

$$v^2 = (G \times M) / r$$

where v is the speed of the orbiting object (the magnitude of its velocity vector)

G is the Universal Gravitational Constant

M is the mass of the body being orbited, in this case the Earth

Take the square root of both sides to isolate the orbital speed, v:

$$v = \text{sqrt}((G \times M) / r)$$

Notice that the orbital speed, v, is a function of the radius of the orbit, r, and two physical constants, G and M. Note also that this relationship is valid only for circular orbits.

Derivation of centripetal acceleration formula

OK, what if we don't start with being given an expression for centripetal force in uniform circular motion, in other words, how was the initial equation above derived:

$$F_{centripital} = m \times a_{centripetal} = (m \times v^2)/r$$

where: m is the mass of the satellite
v is the speed of the satellite and
r is the radius of the circular orbit

The derivation is best made using a diagram of the situation described below. I'm going to use words instead, and give you a link for an explanation with diagram.

Imagine an overhead view of an object in uniform circular motion. The velocity vector of that object is always tangential to the orbit circle at the object's location. Its magnitude of the velocity vector (its speed) does not change as the object orbits, but the orientation of the vector does, since it is always tangent to the orbit at the object's location. We can perform a vector addition on the tangential vectors for the object at two different positions on the orbit, and end up with two triangles that are mathematically similar. One would have the orbit center and the two object positions as its corner points. The other would have the two velocity vectors, moved so that their origins are from the same point. That triangles three corners are the ends of both velocity vectors and their common origin. If we select a time, Δt, between the object being at the two positions short enough so that the object moved less than a quarter of the orbit's circumference, it'll be easier to visualize. The two triangles resulting are isosceles and are mathematically similar – if the object went through 30° of its orbit, say, the angle between the two tangential speed vectors would also be 30°, since both vectors are perpendicular to their respective radius vectors.

For the vector triangle, the vectors are straight, both tangent velocity vectors and their sum vector, Δv. For the position triangle, the two sides of the triangle adjacent to the 30° vertex are of length r, but the opposite side is an arc of the circle, of length v x Δt. Because they are similar triangles, the ratio of their adjacent sides to the opposite side is the same:

$$\Delta v / v = (v \times \Delta t) / r$$

This can be re-arranged as follows. Multiply both sides by v and divide both sides by Δt:

$$\Delta v / \Delta t = v^2 / r$$

dv/dt is always an acceleration, in this case, the centripetal acceleration necessary for uniform circular motion:

$$a_{centripetal} = v^2 / r$$

Which is where we started, with Newton!

I told you a diagram would be useful. I have searched a number of sources, and the best I found was my old physics text: Halliday, David and Resnick, Robert, 1974, *Fundamentals of Physics*, revised printing, New York: John Wiley and Sons, section 4-4, page 48-49. I am certain the text, but not the physics, has been superseded. The best modern reference on this topic I found is: http://www.met.reading.ac.uk/pplato2/h-flap/phys2_6.html.

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